Complex plane: algebraic and geometric properties.

Tuesday, September 12, 2023 9:01 AM

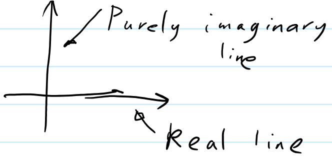
Complex numbers as vectors. Addition and multiplication

$z = \underbrace{a+ib} \rightsquigarrow \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}$ $\quad i \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 1 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Notation: $a = \operatorname{Re} z$
$b = \operatorname{Im} z$

$\mathbb{R}^2$ with addition and multiplication.

$\mathbb{R}$ identified with $\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$.



Addition: usual vector addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

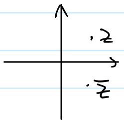
$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac-bd \\ ad+bc \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Notation: $\mathbb{C} := \mathbb{R}^2$ with addition and multiplication.

Absolute value and conjugate

Absolute value and conjugate

$z = a+ib \qquad \bar{z} = a-ib$ - conjugate.



$\mathrm{Re}\, z = \frac{z+\bar{z}}{2}, \quad \mathrm{Im}\, z = \frac{z-\bar{z}}{2i} = -\frac{i}{2}(z-\bar{z}).$

An example: Line $ax+by=c$ $\quad x = \frac{z+\bar{z}}{2}$ $\quad \overset{\mu}{\boxed{\frac{a-ib}{2}}} z + \frac{a+ib}{2}\bar{z} = c$

$a^2+b^2 \neq 0$ $\quad y = \frac{z-\bar{z}}{2i}$

$\boxed{\mu z + \bar{\mu}\bar{z} = c \qquad \boxed{\mu \neq 0}}$ $c \in \mathbb{R}$

Real notation.

Properties 1) $\overline{z+w} = \bar{z}+\bar{w}$ } just computation
2) $\overline{zw} = \bar{z}\,\bar{w}$

Absolute value:

$|z|^2 = x^2+y^2 = (x+iy)(x-iy) = z\bar{z} = |\bar{z}|^2$

Properties: 1. $|zw| = |z||w|$
2. $|z+w| \leq |z|+|w|$

Proof. 1.

$|zw|^2 = zw\overline{zw} = z\bar{z}\cdot w\bar{w} = |z|^2|w|^2.$

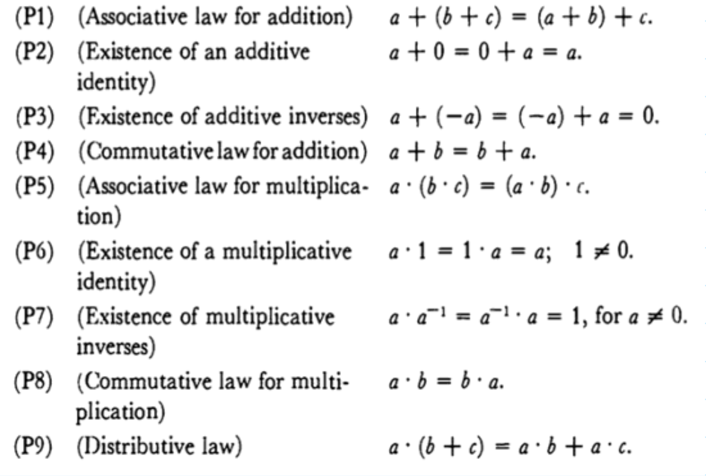
2. The usual triangle inequaity

Complex proof: $|z+w|^2 = (z+w)\left(\overline{z+w}\right) = |z|^2+|w|^2+z\bar{w}+\bar{z}w = |z|^2+|w|^2+2Re(z\bar{w}))$
$\leq |z|^2+|w|^2 + 2|z||w| = (|z|+|w|)^2$

$Re(z\bar{w}) \leq |z\bar{w}| = |z||w|$, By 1.

Notations : $B(z,\delta) = \{w : |z-w| < \delta\}$ - open
$\overline{B}(z,\delta) = \{w : |z-w| \leq \delta\}$ - closed balls centered at z, radius δ.

Complex numbers form a field.



(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c.$
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a.$
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0.$
(P4)	(Commutative law for addition)	$a + b = b + a.$
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$.
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a.$
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c.$

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \Longleftarrow z\overline{z} = |z|^2$$

$$\frac{w}{z} = \frac{w\overline{z}}{|z|^2}$$

Matrix form of a Complex Number.

Fix $z = a+ib$. Map $w = x+iy \to z(x+iy)$ in vector form :

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$M_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad M_z \vec{w} = zw$$

More over: $M_{z_1} + M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & -b_1-b_2 \\ b_1+b_2 & a_1+a_2 \end{pmatrix} = M_{z_1+z_2}$

$$M_{z_1} \cdot M_{z_2} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & a_2b_1 + a_1b_2 \\ -a_2b_1 - a_1b_2 & a_1a_2 - b_1b_2 \end{pmatrix} = M_{z_1 z_2}$$

Let $\mathcal{M} := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$.

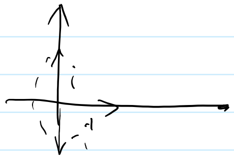
Then $\varphi : \mathbb{C} \to \mathcal{M}$, $\varphi(z) := M_z$ – field isomorphism (bijection preserving + and ×).

What is $M_{\bar{z}}$?

M_z^T

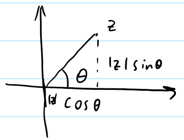
$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $M_z^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar{z}}$.

<u>Remark.</u> $z \to \bar{z}$ – linear map. What is the matrix?



$\Bigg(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\bar{1}} \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\bar{i}} \Bigg)$

Polar form of a Complex Number



z in polar form $:(|z|, \theta)$ θ- angle with $\mathbb{R}_+$.

Real case: $(r,\theta) \longrightarrow \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

Complex notation: $z = |z|(\cos\theta + i\sin\theta)$.

Temporary notation: $\operatorname{cis}\theta := \cos\theta + i\sin\theta$ ($e^{i\theta}$ - later)

$|\operatorname{cis}\theta| = 1$.

z ↗ absolute value $|z|$

↘ $\theta = \arg z$ - not unique. $\arg z = \{\theta + 2\pi k, k \in \mathbb{Z}\} \in \mathbb{R}/_\sim$ $x \sim y \Leftrightarrow x - y = 2\pi k, k \in \mathbb{Z}$.

Principal value of argument: $-\pi < \operatorname{Arg} z \leq \pi$

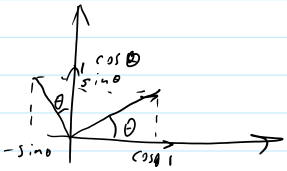
$\arg \bar{z} = -\arg z = \{\theta : -\theta \in \arg z\}$.

$2\arg z$ - well defined, does not depend on representative

$\frac{\arg z}{2}$ - is not! (Does depend on representative)

Rotation as a multiplication.

Rotation of $\mathbb{R}^2$ - linear map.



Matrix: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = M_{\text{cis}\,\theta}$

$\text{cis}\,\theta \cdot w = M_{\text{cis}\,\theta} \vec{w}$ - w rotated by θ. $\boxed{\arg(\text{cis}\,\theta \cdot w) = \arg w + \theta}$

$z = |z| \,\text{cis}(\arg z)$. $zw = |z| \,\text{cis}(\arg z)\, w$. zw - w rotated by arg z dilated by $|z|$.

$\forall \theta_1, \theta_2 \in \arg z$, $\text{cis}\,\theta_1 = \text{cis}\,\theta_2$.

Theorem. 1) $\arg zw = \arg z + \arg w$, $|zw| = |z||w|$.

2) $\arg \frac{z}{w} = \arg z - \arg w$, $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$
$w \neq 0$

(arg z + arg w - take $\theta_1 \in \arg z$, $\theta_2 \in \arg w$ $\arg z + \arg w = \{\theta_1 + \theta_2 + 2\pi k\}$ does not depend on θ_1, θ_2! Prove it!)

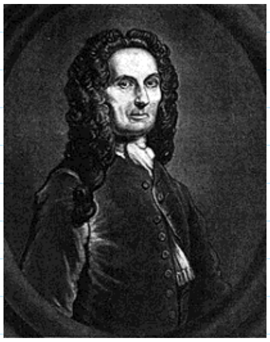
Proof. 1) just done

2) $\arg \frac{1}{w} = \arg \frac{\overline{w}}{|w|^2} = -\arg w$ Another way: $(\arg w + \arg \frac{1}{w} = \arg 1)$

N.B. $\text{Arg}\, zw = \text{Arg}\, z + \text{Arg}\, w$ - $\boxed{\text{not always!}}$ Example?

$z = w = -i$ $\quad \text{Arg}\, z = \text{Arg}\, w = -\frac{\pi}{2}$

$zw = -1$ $\quad \text{Arg} -1 = \pi \neq -\frac{\pi}{2} + \left(-\frac{\pi}{2}\right)$.



Abraham de Moivre

Trigonometry done right: deMoivre formula

$$\mathrm{cis}(\theta_1+\theta_2) = \mathrm{cis}\,\theta_1 \cdot \mathrm{cis}\,\theta_2 \qquad \boxed{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)}$$

de Moivre formula

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$$

$$\cos n\theta = \frac{1}{2}\left(\mathrm{cis}^n\theta + \overline{\mathrm{cis}^n\theta}\right) = \frac{1}{2}\left((\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n\right) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k C_n^{2k}\sin^{2k}\theta\cos^{n-2k}\theta$$

$$\sin n\theta = -\frac{i}{2}\left((\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n\right) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k C_n^{2k+1}\sin^{2k+1}\theta\cos^{n-1-2k}\theta$$

Powers and Roots

Integer powers: $z^n = |z|^n\,\mathrm{cis}(n\arg z).\ n\in\mathbb{N}, n\in\mathbb{Z}$

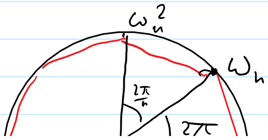
$$i^{239} = 1\cdot\mathrm{cis}\left(239\cdot\frac{\pi}{2}\right) = \mathrm{cis}\,\frac{3\pi}{2} = -i.$$

$$(1+i)^{239} = \underbrace{(\sqrt{2})^{239}}\,\underbrace{\mathrm{cis}\left(239\frac{\pi}{4}\right)} = \sqrt{2}\cdot 2^{119}\cdot\underbrace{\mathrm{cis}\frac{7}{4}\pi} = \boxed{2^{119}(1-i)}.$$

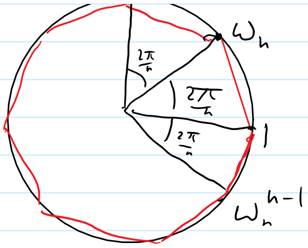
Roots of 1. $z^n=1.\ \begin{cases} n\,\mathrm{Arg}\,z \in \{2\pi k,\ k\in\mathbb{Z}\}. \\ |z|=1 \end{cases}$

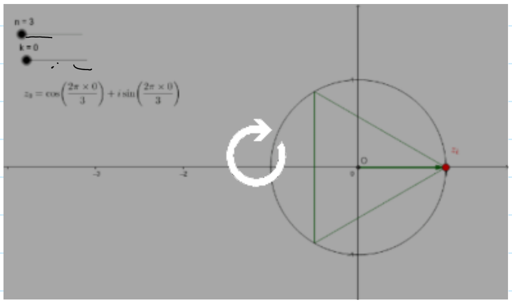
$\omega_n := \mathrm{cis}\left(\frac{2\pi}{n}\right).$ Solutions: $1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}.$

$$\arg\omega_n^{n-1} = \left\{\underbrace{\frac{2\pi(n-1)}{n}} + 2\pi k\right\}$$



Roots of Unity





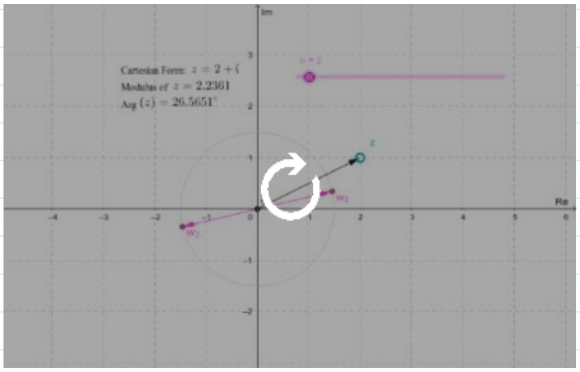
nth roots of $w \neq 0$: $z^n = w$.

z_1, z_2 – two roots. Then $\left(\frac{z_1}{z_2}\right)^n = \frac{w}{w} = 1$. So $\frac{z_1}{z_2}$ - root of 1!

$\Rightarrow$: n roots. $z_0, z_0\omega_n, \dots, z_0\omega_n^{n-1}$

$|z_0| = |w|^{1/n}$. $n \operatorname{Arg} z_0 \in \arg w$.

nth Roots of Complex Numbers



$w = |w|(\cos\varphi + i\sin\varphi)$

Can take

$z_0 = |w|^{1/n}\left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right)$